

RANK 2 QUASIPARABOLIC VECTOR BUNDLES ON \mathbb{P}^1 AND THE VARIETY OF LINEAR SUBSPACES CONTAINED IN TWO ODD-DIMENSIONAL QUADRICS

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Consider $2g + 1$ distinct points $p_1, \dots, p_{2g+1} \in \mathbb{P}^1$, where $g \geq 2$. We recall that a rank 2 quasiparabolic vector bundle on the marked curve $(\mathbb{P}^1, p_1, \dots, p_{2g+1})$ is a rank 2 vector bundle V on \mathbb{P}^1 , with the additional data of a one-dimensional subspace F_j of the fiber V_{p_j} of V over p_j , for every $j = 1, \dots, 2g + 1$. The notion of stability for quasiparabolic bundles usually depends on the choice of some weights. In this paper we will only consider stability with respect to the weights $\{0, \frac{1}{2}\}$ at each marked point, and we will say that a quasiparabolic vector bundle is stable if it stable with respect to these weights (see §1.3 and references therein). By [MS80], there is a fine, projective moduli space \mathcal{N} of stable quasiparabolic vector bundles of rank 2 and degree zero on $(\mathbb{P}^1, p_1, \dots, p_{2g+1})$.

The purpose of this note is to show that \mathcal{N} is isomorphic to the variety of $(g - 2)$ -dimensional linear subspaces of \mathbb{P}^{2g} contained in the intersection of two quadrics.

Theorem. *Let $p_1, \dots, p_{2g+1} \in \mathbb{P}^1$ be distinct points, with $g \geq 2$; assume that $p_j = (\lambda_j : 1)$ with $\lambda_j \in k$. Consider \mathbb{P}^{2g} with homogeneous coordinates $(x_1 : \dots : x_{2g+1})$, and let Q_1 and Q_2 denote the following quadrics:*

$$Q_1: \sum_{j=1}^{2g+1} x_j^2 = 0, \quad Q_2: \sum_{j=1}^{2g+1} \lambda_j x_j^2 = 0.$$

Then the moduli space \mathcal{N} of stable quasiparabolic vector bundles of rank 2 and degree zero on $(\mathbb{P}^1, p_1, \dots, p_{2g+1})$ is isomorphic to the variety \mathcal{G} of $(g - 2)$ -dimensional linear subspaces of \mathbb{P}^{2g} , contained in $Q_1 \cap Q_2$.

Notice that the choice of the degree is not relevant here, as for every $d \in \mathbb{Z}$ \mathcal{N} is isomorphic to the moduli space of stable quasiparabolic vector bundles of rank 2 and degree d on $(\mathbb{P}^1, p_1, \dots, p_{2g+1})$, see §1.3.

The proof of this result relies on two well-known facts. The first is the relation between quasiparabolic vector bundles on \mathbb{P}^1 and invariant vector bundles on hyperelliptic curves, established by Bhosle [BD84]. The second ingredient is the description by Bhosle and Ramanan [DR76] of the moduli space of stable vector bundles on a hyperelliptic curve of genus g , with rank 2 and fixed determinant of odd degree, as the variety of $(g - 2)$ -dimensional linear subspaces of \mathbb{P}^{2g+1} contained in the intersections of two quadrics.

Let us notice that the variety \mathcal{G} is a remarkable example of Fano variety; it has dimension $2g - 2$, Picard number $\rho_{\mathcal{G}} = 2g + 2$, and $-K_{\mathcal{G}}$ very ample (see §1.5). Both varieties \mathcal{N} and \mathcal{G} have been studied in several papers, see

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for instance [Bau91, Bis98, Abe04, Muk05, BHK10] for \mathcal{N} , and §1.5 and references therein for \mathcal{G} .

The moduli space \mathcal{N} has a rich birational geometry: it has been shown by Bauer [Bau91] that it is a small modification of the blow-up of \mathbb{P}^{2g-2} in $2g+1$ points, see §3.1. In particular, we deduce that \mathcal{G} is a rational variety.

Summing-up, we have three different descriptions for the same Fano variety: an embedded description in a grassmannian, a modular description via quasiparabolic vector bundles on \mathbb{P}^1 , and a birational description as the unique Fano small modification of the blow-up of \mathbb{P}^{2g-2} in $2g+1$ points.

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1. PRELIMINARIES

1.1. Notations. If V is a vector bundle over a curve C , we denote by V_p the fiber of V over $p \in C$. Moreover, if $\alpha: V' \rightarrow V$ is a homomorphism between locally free sheaves, we denote by $\alpha_p: V'_p \rightarrow V_p$ the induced linear map.

We denote by $\text{Gr}(r, \mathbb{P}^n)$ the grassmannian of r -dimensional linear subspaces of \mathbb{P}^n .

If \mathcal{M} is a moduli space, we denote by $[E]$ the point of \mathcal{M} corresponding to the object E .

We work over an algebraically closed field k of characteristic zero.

1.2. Quasiparabolic vector bundles. Let C be a smooth projective curve with distinct marked points $p_1, \dots, p_r \in C$. A rank 2 quasiparabolic vector bundle on (C, p_1, \dots, p_r) is given by (V, F_1, \dots, F_r) , where V is a rank 2 vector bundle on C , and F_j is a one-dimensional subspace of V_{p_j} for every $j = 1, \dots, r$.

Let us describe a well-known construction for shifting degrees of quasiparabolic vector bundles (see [MS80, Rem. 5.4] and [Muk03, §12.5]). Given (V, F_1, \dots, F_r) , consider the natural sheaf map

$$\beta: V \longrightarrow \bigoplus_{j=1}^r (V_{p_j}/F_j) \otimes \mathcal{O}_{p_j},$$

and set $V' := \ker \beta$, so that V' is the subsheaf of sections s of V such that $s(p_j) \in F_j$ for all j . We have an exact sequence of sheaves on C :

$$0 \longrightarrow V' \xrightarrow{\alpha} V \xrightarrow{\beta} \bigoplus_{j=1}^r (V_{p_j}/F_j) \otimes \mathcal{O}_{p_j} \longrightarrow 0.$$

Then V' is locally free of rank 2, $\det V' \cong \det V \otimes \mathcal{O}_C(-p_1 - \dots - p_r)$ (hence $\deg V' = \deg V - r$), and α is an isomorphism outside p_1, \dots, p_r , while we have $\operatorname{Im} \alpha_{p_j} = F_j$ and $\dim \ker \alpha_{p_j} = 1$ for every $j = 1, \dots, r$. Thus we get a new rank 2 vector bundle V' on C , with a quasiparabolic structure at the points p_1, \dots, p_r given by the one-dimensional subspaces $F'_j := \ker \alpha_{p_j}$.

Starting from (V', F'_1, \dots, F'_r) , we can repeat the procedure and get a new exact sequence

$$0 \longrightarrow V'' \xrightarrow{\alpha'} V' \xrightarrow{\beta'} \bigoplus_{j=1}^r (V'_{p_j}/F'_j) \otimes \mathcal{O}_{p_j} \longrightarrow 0$$

and a new rank 2 quasiparabolic vector bundle $(V'', F''_1, \dots, F''_r)$. Then V'' is the subsheaf of sections s of V vanishing at all p_j 's, namely $V'' \cong V \otimes \mathcal{O}_C(-p_1 - \dots - p_r)$, and the subspaces F''_j correspond to F_j under this isomorphism. This shows that the quasiparabolic vector bundles (V, F_1, \dots, F_r) and (V', F'_1, \dots, F'_r) determine each other.

1.3. Moduli of stable quasiparabolic vector bundles. A rank 2 quasiparabolic vector bundle $(V, F_1, \dots, F_{2g+1})$ on $(\mathbb{P}^1, p_1, \dots, p_{2g+1})$ is *stable* (with respect to the weights $\{0, \frac{1}{2}\}$ at each p_j) if for every line subbundle $L \subset V$ we have:

$$\# \{j \in \{1, \dots, 2g+1\} \mid L_{p_j} = F_j\} < \deg V + g - 2 \deg L + \frac{1}{2}$$

(see [Muk03, Def. 12.45 and Def. 12.55]; in Mukai's notation, the weight at each point is $\frac{1}{2}$).

Notice that as the right-hand side is not an integer, we can never get equality above; this depends on the fact that the number of marked points is odd, and corresponds to the non-existence of strictly semistable quasiparabolic vector bundles in our setting.

There exists a smooth, projective, fine moduli space \mathcal{N}_d for stable quasiparabolic vector bundles of degree d and rank 2 on $(\mathbb{P}^1, p_1, \dots, p_{2g+1})$ [MS80].

As usual, by tensoring with a line bundle, one sees that $\mathcal{N}_d \cong \mathcal{N}_{d'}$ when d and d' have the same parity. Moreover, using the construction described in §1.2 (see also [Muk03, Lemma 12.51]), we also get $\mathcal{N}_d \cong \mathcal{N}_{d-2g-1}$. We conclude that the moduli spaces \mathcal{N}_d are isomorphic for all d ; in the rest of the paper we will set $d = 0$ and just write \mathcal{N} for \mathcal{N}_0 . In this case the stability condition becomes:

$$(1.4) \quad \# \{j \in \{1, \dots, 2g+1\} \mid L_{p_j} = F_j\} \leq g - 2 \deg L$$

for every line subbundle $L \subset V$.

1.5. The variety \mathcal{G} of $(g-2)$ -dimensional linear subspaces contained in two general quadrics in \mathbb{P}^{2g} . Let $Z \subset \mathbb{P}^{2g}$ be the complete intersection of two quadrics. It is well-known (see [Rei72, Prop. 2.1] or [Dol12, Lemma 8.6.1]) that Z is smooth if and only if, up to a projective transformation of \mathbb{P}^{2g} , we have $Z = Q_1 \cap Q_2$ where Q_1 and Q_2 are the quadrics:

$$Q_1: \sum_{j=1}^{2g+1} x_j^2 = 0 \quad \text{and} \quad Q_2: \sum_{j=1}^{2g+1} \lambda_j x_j^2 = 0,$$

with $\lambda_j \in k$ all distinct.

Let us assume that Z is indeed smooth, and let \mathcal{G} be the variety of $(g-2)$ -dimensional linear subspaces contained in Z . Then \mathcal{G} is smooth, connected, and has dimension $2g-2$ (see [Rei72, Th. 2.6] for smoothness, and [Bor90, Th. 4.1] or [DM98, Th. 2.1] for connectedness). Moreover $-K_{\mathcal{G}}$ is given by the restriction to \mathcal{G} of $\mathcal{O}(1)$ on the grassmannian $\mathrm{Gr}(g-2, \mathbb{P}^{2g})$ (see [Bor90, Rem. 4.3] or [DM98, Rem. 3.2(2)]), in particular \mathcal{G} is a Fano variety, $-K_{\mathcal{G}}$ is very ample, and the Plücker embedding of $\mathrm{Gr}(g-2, \mathbb{P}^{2g})$ in $\mathbb{P}^{\binom{2g+1}{g-1}-1}$ yields an anticanonical embedding for \mathcal{G} . Finally we have $\rho_{\mathcal{G}} = 2g+2 = \dim \mathcal{G} + 4$ [Jia12, Prop. 3.2].

2. PROOF OF THE THEOREM

2.1. As a first step, we embed \mathcal{G} in the variety \mathcal{G}' of $(g-2)$ -dimensional linear subspaces of \mathbb{P}^{2g+1} , contained in the intersection of two $(2g)$ -dimensional quadrics.

Consider \mathbb{P}^{2g+1} with homogeneous coordinates $(x_1 : \dots : x_{2g+2})$. We identify \mathbb{P}^{2g} with the hyperplane $H := \{x_{2g+2} = 0\} \subset \mathbb{P}^{2g+1}$, and $\mathrm{Gr}(g-2, \mathbb{P}^{2g})$ with the subvariety

$$\{[L] \in \mathrm{Gr}(g-2, \mathbb{P}^{2g+1}) \mid L \subset H\}.$$

Fix $\lambda_{2g+2} \in k$ different from $\lambda_1, \dots, \lambda_{2g+1}$, and consider the two quadrics in \mathbb{P}^{2g+1} :

$$Q'_1: \sum_{j=1}^{2g+2} x_j^2 = 0 \quad \text{and} \quad Q'_2: \sum_{j=1}^{2g+2} \lambda_j x_j^2 = 0.$$

Finally let $\mathcal{G}' \subset \mathrm{Gr}(g-2, \mathbb{P}^{2g+1})$ be the variety of $(g-2)$ -dimensional linear subspaces of \mathbb{P}^{2g+1} contained in $Q'_1 \cap Q'_2$; we have

$$\mathcal{G} = \{[L] \in \mathcal{G}' \mid L \subset H\}.$$

Let us consider the involutions of \mathbb{P}^{2g+1} and of \mathcal{G}' given by:

$$(2.2) \quad \begin{aligned} i_{\mathbb{P}^{2g+1}}(x_1 \cdots : x_{2g+2}) &= (x_1 : \cdots : x_{2g+1} : -x_{2g+2}), \\ i_{\mathcal{G}'}([L]) &= [i_{\mathbb{P}^{2g+1}}(L)]. \end{aligned}$$

If L is a linear subspace of \mathbb{P}^{2g+1} , an elementary computation shows that $i_{\mathbb{P}^{2g+1}}(L) = L$ if and only if either $L \subseteq H = \{x_{2g+2} = 0\}$, or L contains the point $(0 : \dots : 0 : 1)$. Since this point is not contained in the quadric Q'_1 , we deduce that \mathcal{G} is the fixed locus of the involution $i_{\mathcal{G}'}$.

2.3. Set $p_{2g+2} := (\lambda_{2g+2} : 1) \in \mathbb{P}^1$, and notice that the points p_1, \dots, p_{2g+2} are distinct. Let $\pi: X \rightarrow \mathbb{P}^1$ be the double cover of \mathbb{P}^1 ramified over p_1, \dots, p_{2g+2} , so that X is a hyperelliptic curve of genus g . Set $w_j := \pi^{-1}(p_j)$ for $j = 1, \dots, 2g+2$, and let $i: X \rightarrow X$ be the hyperelliptic involution.

Let \mathcal{M} be the moduli space of stable rank 2 vector bundles on X , with determinant $\mathcal{O}_X(-w_1 - \dots - w_{2g+1})$; by [DR76] there exists an isomorphism

$$\varphi: \mathcal{M} \longrightarrow \mathcal{G}'.$$

2.4. A vector bundle E on X is *i-invariant* if $i^*E \cong E$. As $i^*\mathcal{O}_X(-w_1 - \dots - w_{2g+1}) \cong \mathcal{O}_X(-w_1 - \dots - w_{2g+1})$, i induces an involution $i_{\mathcal{M}}$ of \mathcal{M} by sending $[E]$ to $[i^*E]$. We denote by $\mathcal{M}^{\mathrm{inv}}$ the locus of *i*-invariant vector bundles in \mathcal{M} , namely the fixed locus of $i_{\mathcal{M}}$.

Set $D := w_1 + \cdots + w_{2g+1}$ and $\beta := \mathcal{O}_X(D)$. In the notation of [DR76, p. 161] we have $i_{\mathcal{M}} = i_{\beta}$, where i_{β} is the involution of \mathcal{M} defined by $i_{\beta}([E]) = [i^*E \otimes \mathcal{O}_X(-w_1 - \cdots - w_{2g+1}) \otimes \beta]$.

As in [DR76, Lemma 2.1], the line bundle β corresponds to a partition $\{1, \dots, 2g+2\} = S \cup T$, where

$$S := \{j \mid \text{the coefficient of } w_j \text{ in } D \text{ is odd}\} = \{1, \dots, 2g+1\},$$

$$T := \{j \mid \text{the coefficient of } w_j \text{ in } D \text{ is even}\} = \{2g\}.$$

Notice that by choosing another divisor D' linearly equivalent to D , and with support contained in $\{w_1, \dots, w_{2g+2}\}$, we get the same partition, with at most S and T interchanged.

By [DR76, Corollary, p. 161] $i_{\mathcal{M}}$ corresponds, under the isomorphism φ , to the involution of \mathcal{G}' induced by the involution of \mathbb{P}^{2g+1} which changes the sign of the coordinates x_j for $j \in S$. This is precisely the involution $i_{\mathcal{G}'}$ in (2.2).

We conclude that φ restricts to an isomorphism between \mathcal{M}^{inv} and \mathcal{G} ; in particular, \mathcal{M}^{inv} is smooth, irreducible, and has dimension $2g-2$ (see §1.5). We are left to show that \mathcal{M}^{inv} is isomorphic to \mathcal{N} .

2.5. The isomorphism $\mathcal{M}^{\text{inv}} \cong \mathcal{N}$ follows basically from [BD84, Prop. 1.2] (see also [Bho90, Prop. 3.2]); we report the details for the reader's convenience.

We first describe a set-theoretical map from \mathcal{N} to \mathcal{M}^{inv} .

Let $(V, F_1, \dots, F_{2g+1})$ be a rank 2 stable quasiparabolic vector bundle on $(\mathbb{P}^1, p_1, \dots, p_{2g+1})$, of degree zero. Its pull-back π^*V inherits the quasiparabolic structure $(\pi^*V, \pi^*F_1, \dots, \pi^*F_{2g+1})$ on the curve X with marked points w_1, \dots, w_{2g+1} . As described in §1.2, we have an exact sequence of sheaves on X :

$$0 \longrightarrow E \xrightarrow{\alpha} \pi^*V \xrightarrow{\beta} \bigoplus_{j=1}^{2g+1} (\pi^*V_{w_j} / \pi^*F_j) \otimes \mathcal{O}_{w_j} \longrightarrow 0,$$

where E is locally free of rank 2, $\det E \cong \mathcal{O}_X(-w_1 - \cdots - w_{2g+1})$, and α is an isomorphism outside w_1, \dots, w_{2g+1} , while $\text{Im } \alpha_{w_j} = \pi^*F_j$ and $\dim \ker \alpha_{w_j} = 1$ for every $j = 1, \dots, 2g+1$.

The natural isomorphism $\pi^*V \cong i^*\pi^*V$ and the exact sequence above induce a natural isomorphism

$$\xi: E \longrightarrow i^*E$$

such that $i^*(\xi) \circ \xi = \text{Id}_E$ (ξ can be thought as a lifting of the involution i to the total space of the vector bundle E); in particular, for every $j = 1, \dots, 2g+2$ we have an induced involution $\xi_{w_j}: E_{w_j} \rightarrow E_{w_j}$.

As the homomorphism β is trivial in w_{2g+2} , $\xi_{w_{2g+2}}$ is the identity. A local computation shows that for $j = 1, \dots, 2g+1$, ξ_{w_j} has eigenvalues 1 and -1 , and the (-1) -eigenspace is $\ker \alpha_{w_j}$.

2.6. Let us show that E is stable (see also [BD84, Prop. 1.2]). By contradiction, suppose the contrary. Then there exists a line subbundle $L \subset E$ such that

$$\deg L \geq \frac{\deg E}{2} = -g - \frac{1}{2}.$$

Since a rank 2 vector bundle has at most one destabilising line bundle (see [Muk03, Prop. 10.38]), we must have $\xi(L) = i^*L \subset i^*E$.

Let $M \subset \pi^*V$ be the line subbundle generated by the image of L under $\alpha: E \rightarrow \pi^*V$. Since $\xi(L) = i^*L$, we must have $\widehat{\xi}(M) = i^*M \subset i^*\pi^*V$ under the natural isomorphism $\widehat{\xi}: \pi^*V \rightarrow i^*\pi^*V$. This implies that $M = \pi^*(M')$, where M' is a line subbundle of V .

Set $J := \{j \in \{1, \dots, 2g+1\} \mid L_{w_j} = \ker \alpha_{w_j}\}$ and $J^c := \{1, \dots, 2g+1\} \setminus J$. Then $L \cong M \otimes \mathcal{O}_X(-\sum_{j \in J} w_j)$, thus $\deg L = \deg M - |J| = 2 \deg M' - |J|$, which yields

$$|J^c| = 2g + 1 - |J| = 2g + 1 + \deg L - 2 \deg M' \geq g + \frac{1}{2} - 2 \deg M'.$$

Notice that if $j \in J^c$, then $M_{w_j} = \text{Im } \alpha_{w_j}$, hence $M'_{p_j} = F_j$. Thus the equation above contradicts the stability of $(V, F_1, \dots, F_{2g+1})$ (see (1.4)).

Therefore E is stable, and $[E] \in \mathcal{M}^{\text{inv}}$.

2.7. The construction in 2.5 can be made in families, starting from the universal family over \mathcal{N} ; this yields a morphism

$$\psi: \mathcal{N} \longrightarrow \mathcal{M}^{\text{inv}}.$$

As \mathcal{N} and \mathcal{M}^{inv} are smooth, irreducible varieties of the same dimension, to conclude that ψ is an isomorphism it is enough to show that ψ is injective.

Let $[(V, F_1, \dots, F_{2g+1})]$ and $[(\tilde{V}, \tilde{F}_1, \dots, \tilde{F}_{2g+1})]$ be two points of \mathcal{N} , with the same image $[E] \in \mathcal{M}^{\text{inv}}$ under ψ . By construction we have two isomorphisms $\xi, \tilde{\xi}: E \rightarrow i^*E$ such that

$$(2.8) \quad i^*(\xi) \circ \xi = \text{Id}_E, \quad i^*(\tilde{\xi}) \circ \tilde{\xi} = \text{Id}_E,$$

$$(2.9) \quad \text{and} \quad \xi_{w_{2g+2}} = \tilde{\xi}_{w_{2g+2}} = \text{Id}_{E_{w_{2g+2}}}.$$

As E is stable, it has only constant automorphisms, and there exists a non-zero constant λ such that $\tilde{\xi} = \lambda \xi$. Then (2.8) implies $\lambda = \pm 1$, and (2.9) yields $\lambda = 1$, namely $\tilde{\xi} = \xi$.

In particular, the (-1) -eigenspaces of ξ_{w_j} and $\tilde{\xi}_{w_j}$ are the same for all $j = 1, \dots, 2g+1$, thus the quasiparabolic vector bundles

$$(E, \ker \alpha_{w_1}, \dots, \ker \alpha_{w_{2g+1}}) \quad \text{and} \quad (E, \ker \tilde{\alpha}_{w_1}, \dots, \ker \tilde{\alpha}_{w_{2g+1}})$$

coincide. As noticed in §1.2, this shows that the quasiparabolic vector bundles $(\pi^*V, \pi^*F_1, \dots, \pi^*F_{2g+1})$ and $(\pi^*\tilde{V}, \pi^*\tilde{F}_1, \dots, \pi^*\tilde{F}_{2g+1})$ are isomorphic, and hence the same holds for $(V, F_1, \dots, F_{2g+1})$ and $(\tilde{V}, \tilde{F}_1, \dots, \tilde{F}_{2g+1})$.

This shows that ψ is injective, and concludes the proof of the Theorem. \square

Remark 2.10. The Verlinde formula [Muk03, §12.5, in particular Remark 12.54] gives

$$h^0(\mathcal{G}, \mathcal{O}(-K)) = h^0(\mathcal{N}, \mathcal{O}(-K)) = 1 + 4 + 4^2 + \dots + 4^{g-1} = \frac{4^g - 1}{3}.$$

On the other hand, we have $\mathcal{G} \subset \mathbb{P}^{\binom{2g+1}{g-1}-1}$ under the Plücker embedding. Then one can check that $h^0(\mathcal{G}, \mathcal{O}_{\mathcal{G}}(-K_{\mathcal{G}})) = \binom{2g+1}{g-1}$ for $g = 2, 3$, while $h^0(\mathcal{G}, \mathcal{O}_{\mathcal{G}}(-K_{\mathcal{G}})) > \binom{2g+1}{g-1}$ for $g \geq 4$, so that $\mathcal{G} \subset \mathbb{P}^{\binom{2g+1}{g-1}-1}$ is not linearly

normal for $g \geq 4$; see [Küc96, Th. 1 and Th. 3] and [DM98, Rem. 4.2(2)] for related results. Notice that instead, for $g = 2$, $\mathcal{G} \subset \mathbb{P}^4$ is a Del Pezzo surface of degree 4, and is projectively normal (see for instance [Dol12, Th. 8.3.4]).

Remark 2.11 (The case of an even number of marked points). Let \mathcal{N}_d^+ be the moduli space of semistable rank two quasiparabolic vector bundles of degree d on $(\mathbb{P}^1, p_1, \dots, p_{2g+2})$. As in §2.5, one can associate to $(V, F_1, \dots, F_{2g+2})$ an i -invariant vector bundle E on the hyperelliptic curve X ; however, the degree of E is even, so this relates \mathcal{N}_d^+ to the moduli space \mathcal{M}^+ of semistable rank two vector bundles on X with fixed determinant of even degree. Moreover, the resulting map $\mathcal{N}_d^+ \rightarrow \mathcal{M}^+$ is not injective, but has degree two onto its image; for more details we refer the reader to [Kum00, Th. 2.1] and [Abe04, §2.13] and references therein.

3. FINAL REMARKS

3.1. Birational geometry: relation with the blow-up of \mathbb{P}^n at $n+3$ general points. Set $n := 2g-2$, so that $2g+1 = n+3$. Let $q_1, \dots, q_{2g+1} \in \mathbb{P}^n$ be the images of $p_1, \dots, p_{2g+1} \in \mathbb{P}^1$ under the Veronese embedding $\mathbb{P}^1 \hookrightarrow \mathbb{P}^n$; notice that the points q_1, \dots, q_{n+3} are in general position, in the sense that any $n+1$ points are projectively independent.

Let Y be the blow-up of \mathbb{P}^n at q_1, \dots, q_{n+3} . It has been shown by Bauer [Bau91] (see also [Muk03, Th. 12.56]) that there exists a birational map $\mathcal{N} \dashrightarrow Y$, which is an isomorphism in codimension one. More precisely, Y is a Mori dream space, \mathcal{N} is its (unique) Fano small modification, and the birational map $\mathcal{N} \dashrightarrow Y$ factors a sequence of K -negative flips.

The cone of effective divisors $\text{Eff}(Y) \subset \text{Pic}(Y) \otimes \mathbb{R}$ and the Cox ring of Y are described in [Muk01, CT06]; $\text{Eff}(Y)$ has 2^{n+2} extremal rays. Since the Picard group, the cone of effective divisors, and the Cox ring are invariant under the small modification $\mathcal{N} \dashrightarrow Y$, the same description applies as well to \mathcal{N} and \mathcal{G} .

Notice that when $g = 2$, \mathcal{G} is just the intersection of two quadrics in \mathbb{P}^4 (namely a Del Pezzo surface of degree 4), Y is the blow-up of \mathbb{P}^2 at 5 points, and $\mathcal{G} \cong Y$.

3.2. Dimension 4. Let us set $g = 4$ in this paragraph, so that $\mathcal{G} \subset \text{Gr}(1, \mathbb{P}^6)$ is the variety of lines contained in the intersection of two quadrics in \mathbb{P}^6 , and has dimension $n = 4$. The Fano 4-fold \mathcal{G} has been studied by Borcea [Bor91] and Küchle [Küc95]; it has $b_2 = 8$, $b_3 = 0$, $b_4 = h^{2,2} = 30$, $(-K)^4 = 80$, and $h^0(-K) = 21$. It is a peculiar example of Fano 4-fold, because it has “large” second Betti number: the only other examples known to the author of Fano 4-folds with $b_2 \geq 8$ are products of Del Pezzo surfaces.

As above let Y be the blow-up of \mathbb{P}^4 in 7 points. The small modification $Y \dashrightarrow \mathcal{G}$ has a simple, explicit description as a sequence of 22 K -positive flips, see [Muk03, Ex. 12.57].

The cone of effective curves $\text{NE}(\mathcal{G})$ has 64 extremal rays; these are all small, of type $(2, 0)$, with exceptional locus a surface $L \cong \mathbb{P}^2$ with normal bundle $\mathcal{O}(-1)^{\oplus 2}$ [Bor91, Th. 4.3]. These surfaces in fact are given by the lines contained in the 64 planes contained in $Q_1 \cap Q_2 \subset \mathbb{P}^6$.

We conclude by noting that the blow-up of \mathcal{G} at a general point is still a Mori dream space, because it is a small modification of a blow-up of \mathbb{P}^4 at 8 general points, which is a Mori dream space (see [CT06, Th. 1.3] and also [Muk05, §2]). It would be interesting to know whether this blow-up still has a Fano small modification; this would give an example of Fano 4-fold with $b_2 = 9$, which is not a product of surfaces.

REFERENCES

- [Abe04] T. Abe, *Anticanonical divisors of a moduli space of parabolic vector bundles of half weight on \mathbb{P}^1* , Asian J. Math. **8** (2004), 395–408.
- [Bau91] S. Bauer, *Parabolic bundles, elliptic surfaces and $SU(2)$ -representation spaces of genus zero Fuchsian groups*, Math. Ann. **290** (1991), 509–526.
- [BD84] U. N. Bhosle-Desale, *Degenerate symplectic and orthogonal bundles on \mathbb{P}^1* , Math. Ann. **267** (1984), 347–364.
- [BHK10] I. Biswas, Y. I. Holla, and C. Kumar, *On moduli spaces of parabolic vector bundles of rank 2 on \mathbb{CP}^1* , Michigan Math. J. **59** (2010), 467–479.
- [Bho90] U. N. Bhosle, *Pencils of quadrics and hyperelliptic curves in characteristic two*, J. reine angew. Math. **407** (1990), 75–98.
- [Bis98] I. Biswas, *A criterion for the existence of a parabolic stable vector bundle of rank two over the projective line*, Int. J. Math. **9** (1998), 523–533.
- [Bor90] C. Borcea, *Deforming varieties of k -planes of projective complete intersections*, Pacific J. Math. **143** (1990), 25–36.
- [Bor91] ———, *Homogeneous vector bundles and families of Calabi-Yau threefolds. II*, Several Complex Variables and Complex Geometry, Part 2 (Santa Cruz, CA, 1989), Proc. Symp. Pure Math., vol. 52, 1991, pp. 83–91.
- [CT06] A.-M. Castravet and J. Tevelev, *Hilbert’s 14th problem and Cox rings*, Compos. Math. **142** (2006), 1479–1498.
- [DM98] O. Debarre and L. Manivel, *Sur la variété des espaces linéaires contenus dans une intersection complète*, Math. Ann. **312** (1998), 549–574.
- [Dol12] I. V. Dolgachev, *Classical algebraic geometry - a modern view*, Cambridge University Press, 2012.
- [DR76] U. V. Desale and S. Ramanan, *Classification of vector bundles of rank 2 on hyperelliptic curves*, Invent. Math. **38** (1976), 161–185.
- [Jia12] Z. Jiang, *A Noether-Lefschetz theorem for varieties of r -planes in complete intersections*, Nagoya Math. J. **206** (2012), 39–66.
- [Küc95] O. Küchle, *On Fano 4-folds of index 1 and homogeneous vector bundles over Grassmannians*, Math. Z. **218** (1995), 563–575.
- [Küc96] ———, *Some properties of Fano manifolds that are zeros of sections in homogeneous vector bundles over Grassmannians*, Pacific J. Math. **175** (1996), 117–125.
- [Kum00] C. Kumar, *Invariant vector bundles of rank 2 on hyperelliptic curves*, Michigan Math. J. **47** (2000), 575–584.
- [MS80] V. B. Mehta and C. S. Seshadri, *Moduli of vector bundles on curves with parabolic structures*, Math. Ann. **248** (1980), 205–239.
- [Muk01] S. Mukai, *Counterexample to Hilbert’s fourteenth problem for the 3-dimensional additive group*, RIMS Preprint n. 1343, Kyoto, 2001.
- [Muk03] ———, *An introduction to invariants and moduli*, Cambridge Studies in Advances Mathematics, vol. 81, Cambridge University Press, 2003.
- [Muk05] ———, *Finite generation of the Nagata invariant rings in A - D - E cases*, RIMS Preprint n. 1502, Kyoto, 2005.
- [Rei72] M. Reid, *The complete intersection of two or more quadrics*, Ph.D. thesis, University of Cambridge, 1972, available at the author’s webpage homepages.warwick.ac.uk/~masda/3folds/qu.pdf.

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